Binary neutron-star systems: From the Newtonian regime to the last stable orbit

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(Received 30 March 1998; published 22 October 1998)

We report on the first calculations of fully relativistic binary circular orbits to span a range of separation distances from the innermost stable circular orbit (ISCO), deeply inside the strong field regime, to a distance (~ 200 km) where the system is accurately described by Newtonian dynamics. We consider a binary system composed of two identical corotating neutron stars, with 1.43 $M_{\odot}$ gravitational mass each in isolation. Using a conformally flat spatial metric we find solutions to the initial value equations that correspond to semi-stable circular orbits. At large distance, our numerical results agree exceedingly well with the Newtonian limit. We also present a self consistent determination of the ISCO for different stellar masses. [S0556-2821(98)04222-2]

PACS number(s): 04.25.Dm, 04.40.Dg, 95.30.Sf, 97.80.Fk

Neutron-star binary systems are currently of interest as sources of gravitational radiation [1,2]. The development of a new generation of laser interferometric and cryogenic gravitational wave detectors [3] has renewed interest in theoretical models for generating gravitational radiation. Binary systems composed by neutron stars and/or black holes are the most promising sources of detectable gravitational radiation. However, the expected signal-to-noise ratio is so low that the extraction of information will be difficult without good theoretical waveforms. Thus, it is important to accurately model such systems.

At the frequencies of interest for laser interferometric (10–100 Hz) and cryogenic (~ 1000 Hz) detectors the orbits enter the strong gravity regime. Expansion methods applicable in the Newtonian limit begin to breakdown before the strong field regime. Hence, it becomes imperative to model the binary evolution with a method which accurately extends from the Newtonian limit to the strong gravity regime. In this paper we present the first such calculation. These calculations are for one particular scenario, namely that of two equal-mass stars constrained to rigid corotation in circular orbits. Nevertheless, this represents a plausible benchmark for the expected gravity wave signal.

Many groups have performed numerical simulations using different approximations to this problem. Preliminary results have been achieved using Newtonian dynamics [4] or post-Newtonian expansion techniques [5]. While these approximations work well when the stars are very far from each other, they become unreliable when the distance between stars reduces to a few stellar radii.

Recently, a new fully relativistic approximation has been used [6–8] to model binary neutron-star systems. This approximation is mainly based upon a restriction that the spatial part of the metric tensor is forced to be conformally flat. Work by Wilson and Mathews [6] and Wilson, Mathews and Marronetti [7] reported the first fully relativistic calculations of quasi-stable circular orbits employing this approximation. The most controversial [9,19] of their results predicts that, for certain conditions, the stars could collapse into black holes prior to the merger. This controversy is irrelevant to the present work in that it has been demonstrated [8,10] that this effect does not occur when rigid corotation is imposed.

Baumgarte *et al.* [10] recently developed a method for using the conformally flat condition (CFC) to compute rigidly corotating stars. They found circular orbits for very close separation distances (a few stellar radii) and estimated the point of secular instability of the stars. In the present work however, we implement a scheme to directly determine the location of the ISCO for systems with different total masses. We also present numerical solutions for quasi-stable circular orbits for corotating neutron stars that cover a wide range of separation distances between the stars. Hence for the first time we present the much needed calculation connecting the strong field regime at the *innermost stable circular orbit* (ISCO) with the weak field region where the system is well described by Newtonian dynamics.

A full discussion of the CFC method can be found in [7]. We use the (3+1) spacetime slicing as defined in the Arnowitt-Deser-Misner (ADM) formalism [11,12]. Utilizing Cartesian x,y,z isotropic coordinates, proper distance is expressed

$$ds^2 = -(\alpha^2 - \beta_n\beta^n)dt^2 + 2\beta_n dx^n dt + \gamma_{ns} dx^n dx^s, \quad (1)$$

where the lapse function α describes the differential lapse of proper time between two hypersurfaces. The quantity β_i is the shift vector denoting the shift in space-like coordinates between hypersurfaces and γ_{ij} is the spatial three-metric. The Latin indices go from 1 to 3.

Using York's (3+1) formalism [12], the initial value equations can be written as follows; the Hamiltonian constraint equation can be written

$$R = 16\pi\rho + K_{ns}K^{ns} - K^2, \quad (2)$$

where R is the Ricci scalar curvature, K_{ns} is the extrinsic curvature, and ρ is the mass-energy density.

The momentum constraints have the form [13]

$$D_n(K^{ni} - \gamma^{ni}K) = 8\pi S^i, \quad (3)$$

where D_n is the three-space covariant derivative and S^i are the spatial components of the four-momentum density.

The CFC method restricts the spatial metric γ_{ij} to the form

$$\gamma_{ij} = \phi^4 \delta_{ij}, \tag{4}$$

where the conformal factor ϕ is a positive scalar function. This approximation simplifies greatly the equations. It is motivated in part by the fact that the gravitational radiation in most systems studied so far is small compared to the total gravitational mass. The CFC is a frequently employed approach to the initial value problem in numerical relativity. Its application here is consistent with the quasi-equilibrium orbit approximation. Further justification for its use can be found in Refs. [8] and [14].

The CFC leads to a set of elliptic equations for the metric components. Using Eq. (2) in combination with the maximal slicing condition $\mathrm{tr}(K) = 0$, we get the following equations for ϕ and $(\alpha\phi)$:

$$\nabla^2 \phi = -4\pi\rho_1, \tag{5}$$

$$\nabla^2(\alpha\phi) = 4\pi\rho_2, \tag{6}$$

where the ∇_i represent flat-space derivatives and the source terms are

$$\rho_1 = \frac{\phi^5}{2}\left[\rho_0 W^2 + \rho_0\epsilon[\Gamma(W^2-1)+1] + \frac{1}{16\pi}K_{ns}K^{ns}\right] \tag{7}$$

$$\rho_2 = \frac{\alpha\phi^5}{2}\left[\rho_0(3W^2-2) + \rho_0\epsilon[3\Gamma(W^2+1)-5] + \frac{7}{16\pi}K_{ns}K^{ns}\right], \tag{8}$$

where ρ_0 is the rest-mass density, ϵ the internal energy per unit of rest mass, Γ the adiabatic index, and W a generalization of the special relativistic γ factor [7]. A solution of Eq. (6) determines the lapse function after Eq. (5) is used to determine the conformal factor.

The shift vector β^i can be decomposed [15]:

$$\beta^i = B^i - \frac{1}{4}\nabla^i \chi. \tag{9}$$

This is introduced into Eq. (3) to obtain the following two elliptic equations:

$$\nabla^2 \chi = \nabla_n B^n, \tag{10}$$

$$\nabla^2 B^i = 2\nabla_n \ln(\alpha\phi^{-6})K^{in} - 16\pi\alpha\phi^4 S^i. \tag{11}$$

An equation for the extrinsic curvature $\hat{K}^{ij}$ is derived [7] using the time evolution equation and the maximal slicing condition

$$\hat{K}^{ij} = \frac{\phi^6}{2\alpha}(\nabla^i\beta^j + \nabla^j\beta^i - \tfrac{2}{3}\delta_{ij}\nabla_n\beta^n), \tag{12}$$

where $\hat{K}^{ij} = \phi^{10}K^{ij}$.

We assume that the matter behaves like a perfect fluid with a stress-energy tensor

$$T^{\mu\nu} = (\rho_0(1+\epsilon)+P)u^\mu u^\nu + Pg^{\mu\nu}, \tag{13}$$

and use a polytropic equation of state (EOS)

$$P = k\rho_0^\Gamma, \tag{14}$$

with P the pressure and k a constant. The results presented here are for stars with $\Gamma = 2$.

Following Baumgarte *et al.* [10], for rigidly corotating stars, the fluid four velocity can be taken as proportional to a Killing vector

$$\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial\Phi}, \tag{15}$$

where ω is the orbital angular frequency and Φ the azimuth coordinate. In this case the steady-state limit of the hydrodynamics momentum equation [8] yields the relativistic Bernoulli equation [10]

$$q = \frac{1}{1+n}\left(\frac{1+C}{\alpha(1-v^2)^{1/2}} - 1\right), \tag{16}$$

where $q = P/\rho_0$, C is a constant of integration and v is the matter proper velocity [16]. From Eqs. (13) and (16) we find expressions for the proper baryonic matter density ρ_0 and the material momentum density S^i as functions of the fields. We choose α, β^i, ϕ, and q as the independent variables of our set of equations.

The set of equations is solved numerically using an iterative algorithm based upon a specially designed elliptic solver. This method consists of a combination of multigrid algorithms and domain decomposition techniques [17]. The three-dimensional spatial volume where the equations are solved is divided into concentric layers. These layers are centered around the star and the grid resolution decreases with the distance from the stars. To avoid bias, each iteration cycle starts assuming zero angular frequency and spherical stars. The equations are solved iteratively until numerical convergence for the fields and the frequency is achieved. Our calculations utilize 40 grid points on average across the stellar diameter. This is about twice as large as the number used in previous work [10]. This efficient use of computer memory permits us to describe adequately the interior of the stars even when they are so far apart that they approach the regime of Newtonian point masses.

The boundary conditions are estimated from the first terms in a multipole expansion of the fields. This method is quite accurate when the boundary surfaces are very far from the stars, which is the case in our computations. Thus, these methods allow us to connect between the strong gravitational field regime close to the ISCO and the Keplerian regime.

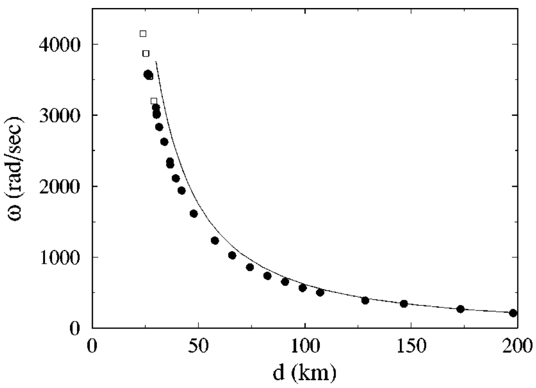

FIG. 1. Orbital frequency as a function of the coordinate separation between star centers. The solid line represents Kepler's law, the circles are the orbits corresponding to a sequence of stellar baryonic mass of 1.55 $M_\odot$, and the squares are results from Baumgarte *et al.* [10].

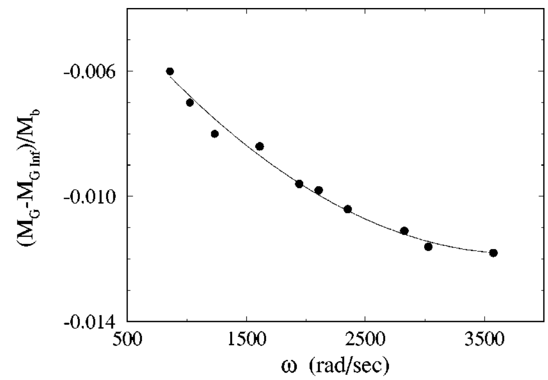

FIG. 2. Binding energy of the system as a function of the orbital frequency as seen by a distant observer. The points represent the orbits of the high resolution runs and the solid line corresponds to a polynomial fit. We identify the lowest point as the ISCO of the system.

Domain decomposition techniques also provide a natural way of code parallelization, which reduces the processing time enormously [17].

Solutions are obtained for specific values of two free parameters, namely, the coordinate distance between stars and the stellar baryonic mass. The reason to choose the latter as a free parameter is that the baryonic mass of the star remains constant during the inspiral that describes most of the time evolution of the system. This differs from the approach of [10] who used central density and separation distances as free parameters and interpolated to find the solutions of constant baryonic mass. In our calculations we iterate to obtain a given baryonic mass. This avoids the numerical error associated with interpolation. Thus, we were able to construct constant baryonic mass sequences of orbits with a minimum number of code runs.

Using the CFC approximation we found solutions to the initial value equations for semi-stable circular orbits for a binary system of identical neutron stars with 1.55 $M_\odot$ baryonic mass and 1.43 $M_\odot$ gravitational mass in isolation. The former values represent a typical neutron star and were obtained by fixing the value of the polytropic constant (in polytropic models physical quantities can be rescaled with the polytropic constant [10]).

Figure 1 shows the angular frequency as seen by a distant observer as a function of the coordinate distance between the centers of the stars (circles). The coordinate distance is not a gauge invariant quantity, but since it converges asymptotically to flat space separations, it allows us to compare our results with Kepler's law (solid line). We can also compare our results with those from Baumgarte *et al.* [10] (squares) since we use the same coordinates. They estimate the onset of secular instability after which the stars can no longer maintain rigid corotation. They assume that the ISCO is close to this point since the secular instability is expected to occur just before the dynamical instability that defines the ISCO [18]. Note that their calculations include numerical solutions for orbits that are inside the ISCO which are used to determine the turning point in the binding energy [10].

The numerical method developed for this work shows the absence of solutions for circular orbits when the stars are too close. This situation is consistent with the existence of a dynamical instability which defines an innermost stable circular orbit for the binary system. However, there is still the possibility that this absence is due to a problem in the convergence of the numerical scheme. The ISCO determined here is thus the stable orbit with the highest angular frequency (in the sequence shown in Fig. 1, it is at $\omega = 3576$ rad/sec).

The value of angular momentum at the ISCO is $J = 1.61 \times 10^{11}$ cm^2, which gives a value of $J/M_G^2 = 0.94$. Thus, the stars could merge into a Kerr black hole without the further loss of angular momentum.

Figure 2 shows the binding energy as a function of the angular frequency of the system as seen by a distant observer. The binding energy is represented as one half the ADM mass of the binary minus the gravitational energy of an isolated star, divided by the baryonic mass of the isolated star. The points correspond to the orbits obtained using high resolution (more than 30 grid points across the stellar diameter) and the solid line corresponds to a polynomial fit. The lowest plotted value of the binding energy on Fig. 2 coincides with the orbit we identify as the ISCO. In this calcula-

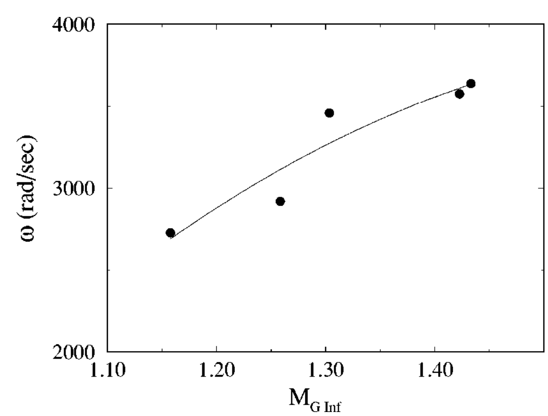

FIG. 3. Orbital angular frequency of the ISCO for stars with different gravitational masses (in units of solar masses) in isolation.

tion no turning point was observed (within the numerical error). For large separations (low angular frequencies) the value of the binding energy approaches zero as expected.

Figure 3 shows the ISCO angular frequency for systems of stars with different gravitational mass in isolation. The angular frequencies shown in Fig. 3 are somewhat lower ($\sim$10%) than the estimations corresponding to the secular instability points from Baumgarte *et al.* [10]. This is most likely due to effects of grid resolution.

The excellent agreement between our results at large separations and the values obtained from Newtonian dynamics provides an additional check on the numerical code.

We observe a slight decrease in the central density when the stars approach each other similar to the results reported in [10]. This is contrary to what is observed in fully hydrodynamical simulations. In Ref. [8] however it has been shown analytically and numerically that no increase in central density occurs for stars in rigid corotation. This also confirms that the effect is not an artifact of the CFC method.

An analysis of the emission of gravitational radiation along sequences of constant baryonic mass will be the subject of a forthcoming article.

Work at University of Notre Dame was supported in part by NSF grant PHY-97-22086. This work was performed in part under the auspices of the U. S. Department of Energy by the Lawrence Livermore National Laboratory under contract W-7405-ENG-48.

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